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Vacuum structure and gravitational bags produced by metric-independent space–time volume-form dynamics

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We propose a new class of gravity-matter theories, describing $R + R^2$ gravity interacting with a nonstandard nonlinear gauge field system and a scalar "dilaton," formulated in terms of two different non-Riemannian volume-forms (generally covariant integration measure densities) on the underlying space–time manifold, which are independent of the Riemannian metric. The nonlinear gauge field system contains a square-root $\sqrt{-F^2}$ of the standard Maxwell Lagrangian which is known to describe charge confinement in flat space–time. The initial new gravity-matter model is invariant under global Weyl-scale symmetry which undergoes a spontaneous breakdown upon integration of the non-Riemannian volume-form degrees of freedom. In the physical Einstein frame we obtain an effective matter-gauge-field Lagrangian of "k-essence" type with quadratic dependence on the scalar "dilaton" field kinetic term X, with a remarkable effective scalar potential possessing two infinitely large flat regions as well as with nontrivial effective gauge coupling constants running with the "dilaton" φ. Corresponding to each of the two flat regions we find "vacuum" configurations of the following types: (i) $\varphi = \text{const}$ and a nonzero gauge field vacuum $\sqrt{-F^2} \neq 0$, which corresponds to a charge confining phase; (ii) $X = \text{const}$ ("kinetic vacuum") and ordinary gauge field vacuum $\sqrt{-F^2} = 0$ which supports confinement-free charge dynamics. In one of the flat regions of the effective scalar potential we also find: (iii) $X = \text{const}$ ("kinetic vacuum") and a nonzero gauge field vacuum $\sqrt{-F^2} \neq 0$, which again corresponds to a charge confining phase. In all three cases, the space–time metric is de Sitter or Schwarzschild–de Sitter. Both "kinetic vacuums" (ii) and (iii) can exist only within a finite-volume space region below a de Sitter horizon. Extension to the whole space requires matching the latter with the exterior region with a nonstandard Reissner–Nordström–de Sitter geometry carrying an additional constant radial background electric field. As a result, we obtain two classes of gravitational bag-like configurations with properties, which on one hand partially parallel some of the properties of the solitonic "constituent quark" model and, on the other hand, partially mimic some of the properties of MIT bags in QCD phenomenology.

Keywords: Modified gravity theories; non-Riemannian volume-forms; global Weyl-scale symmetry spontaneous breakdown; flat regions of scalar potential; charge confining non-linear gauge field system; gravitational bags.

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1. Introduction

To understand the basic features of hadronic physics we are unavoidably lead to the existence of different phases of the gauge theory. Furthermore, these different phases are apparently associated with different values of the vacuum energy density, as described for example in the *MIT bag model* of hadrons.[1,2] Inside the MIT bag, quarks and gluons propagate almost freely and there the vacuum energy density is big, defining the so-called bag constant. Outside the bag there is no propagation of either quarks or gluons, and in the MIT bag model the outside vacuum energy density is set to zero (any choice for the outside energy density is possible, if we ignore gravity, since while ignoring gravity only the difference of the energy densities inside and outside the bag is of significance). For a vacuum state $p = -\rho$, where the vacuum pressure inside the bag is negative while zero outside, an empty bag therefore tends to implode. When the bag is filled with particles, the positive pressure of the particles stabilizes the bubble at a certain radius.

It is interesting that a similar "two phase" structure defined via two vastly different scales of the vacuum energy density does also appear in cosmology. Indeed, according to the most accepted scenario of the early universe — the inflation picture,[3–8] there was at the beginning a large vacuum energy density. Together with this, for a description of the present slowly accelerated phase of the universe[9–12] (for a review, see Ref. 13) one employs a small vacuum energy density.

A scenario of continuously connecting an inflationary phase to a slowly accelerating universe through the evolution of a single scalar field — the so-called "*quintessential inflation*" scenario — has been first proposed in Ref. 14, which triggered active further development (models based on generalized $F(R)$ gravity;[15] based on the k-essence[16–19] framework — see Ref. 20; based on the "variable gravity" mode[21] and containing an extensive list of references to earlier work on the topic — see Refs. 22 and 23).

In the cosmological context we have been able to construct models providing a unified scenario where both an inflation and a slowly accelerated phase for the universe can appear naturally from the existence of two infinitely large flat regions in the effective scalar field potential with vastly different scales which we derive systematically from a Lagrangian action principle.[24,25] Namely, we have constructed a new kind of globally Weyl-scale invariant gravity-matter action within the first-order (Palatini) approach formulated in terms of two different non-Riemannian volume-forms (generally covariant integration measure densities on the pertinent space–time manifold independent of the Riemannian metric). The principal feature is the requirement of global Weyl-scale invariance and the choice of different scaling properties of the two non-Riemannian measures (volume elements) which dictates the precise form of the terms in the action. In this new theory, there is a single scalar field with kinetic terms coupled to both non-Riemannian measures, whereas the standard Einstein–Hilbert term R and the R^2-term couple each to a different non-Riemannian measure.

Global Weyl-scale invariance is spontaneously broken upon solving part of the equations of motion corresponding to the auxiliary antisymmetric tensor gauge fields of maximal rank defining the two non-Riemannian measures — due to the appearance of two arbitrary dimensionful integration constants. The latter produce a remarkable effect on the resulting physical Einstein-frame theory[24,25] — we find there an effective k-essence[16–19] type of theory, where the effective scalar field potential has *two infinitely large flat regions* corresponding to the two accelerating phases of the universe — one for large negative values of the scalar field with a very large height corresponding to the early universe, and another one for large positive values of the scalar field with a very low height corresponding to the universe of the present epoch.

Since the construction was based on geometrical and symmetry considerations, which are very general, one may think that a similar model can be constructed for a different physical application, i.e. the phases of a gauge theory. As we will see here, it is indeed possible to obtain a phase structure of confinement and deconfinement related to what the MIT bag model suggests.

To this end, we will couple the above new type of gravity-matter theory defined in terms of the two different non-Riemannian measures and containing R^2 gravity term to a special kind of nonstandard nonlinear gauge field model (for the analogous situation in the less general case of gravity-matter models with one non-Riemannian and one standard Riemannian integration measures, see Ref. 26).

Namely, let us consider Abelian gauge fields whose Lagrangian contains both the standard Maxwell Lagrangian ($\sim F^2 \equiv F_{\mu\nu}F^{\mu\nu}$) as well as the nonstandard square-root of the latter ($\sim \sqrt{-F^2}$) coupled to the two different non-Riemannian measures in a globally Weyl-scale invariant form. In flat space–time, the $\sqrt{-F^2}$-term is known to describe dynamics of charge confinement[27–31] related to the nonzero vacuum value of $\sqrt{-F^2}$. The latter is an explicit realization of an earlier proposal by 't Hooft[32,33] who argued that the energy density of electrostatic field configurations in the low-energy description of confining quantum gauge theories must be a linear function of the electric displacement field in the infrared region (the latter appearing as a quantum "infrared counterterm"). In App. B we extend the flat space–time proof in Ref. 28 about the charge confining property of the $\sqrt{-F^2}$-term to the case of curved static spherically-symmetric space–times.

For further interesting properties of gravity-matter theories involving the "square-root" Maxwell term $\sqrt{-F^2}$ (black holes with confining electric potential, new mechanism for dynamical generation of cosmological constant, charge-hiding and charge-confining via "tube-like" wormholes), see Refs. 34–36.

Let us remark that one could start with the non-Abelian version of the nonlinear gauge field system with $\sqrt{-\mathrm{Tr}\left(F_{\mu\nu}F^{\mu\nu}\right)}$. Since we will be interested in static spherically symmetric solutions, the non-Abelian theory effectively reduces to an Abelian one as pointed out in Refs. 27–31.

In the present context after including the coupling of the non-Riemannian-measures-modified gravity-matter theory to the nonlinear gauge field with $\sqrt{-F^2}$, in the pertinent physical Einstein-frame we obtain an effective matter Lagrangian again of "k-essence" type with quadratic dependence on the φ kinetic term X of the scalar "dilaton" field φ, with a remarkable effective scalar potential possessing two infinite flat regions with different energy scales. In addition, we get a nontrivial coupling of the nonlinear gauge field to the "dilaton" kinetic term $X\sqrt{-F^2}$. All terms are multiplied by nontrivial "dilaton"-dependent coefficient functions, including nontrivial effective gauge coupling constants running with φ. An important observation is their "flatness" (constancy with respect to running φ) in both infinitely large flat regions of the effective potential.

We study the static spherically symmetric "vacuum" configurations corresponding to each of the two flat regions. In all cases the gravitational part is de Sitter type (de Sitter or Schwarzschild–de Sitter) with effective cosmological constant whose value is determined by the height of the total effective "dilaton" potential in each of the flat regions. The latter includes, apart from the purely scalar "dilatonic" ones, also additional contributions due to (possible) nonzero vacuum values of $\sqrt{-F^2}$ and X.

The static spherically symmetric "vacuum" configurations of the "dilaton" φ and the nonlinear gauge field are of the following types:

(i) $\varphi = \text{const}$, i.e. $X = 0$ and a nonzero gauge field vacuum $\sqrt{-F^2} \neq 0$, the latter corresponding to a confining phase — these solutions exist in both flat regions of the effective scalar potential (the one for large negative values of φ and the other one for large positive values of φ).

(ii) $X = \text{const}$ ("kinetic vacuum") and ordinary gauge field vacuum $\sqrt{-F^2} = 0$, which supports *confinement-free* charge dynamics — this solution exists both in the flat region of the effective scalar potential for large positive values of φ as well as for a special value of one of the scalar potential's parameters also in the flat region for large negative values of φ.

(iii) $X = \text{const}$ ("kinetic vacuum") and a nonzero gauge field vacuum $\sqrt{-F^2} \neq 0$, which again corresponds to a confining phase — this solution exist in the flat region of the effective scalar potential for large negative values of φ for generic scalar potential's parameter values.

An important point here is that both "kinetic vacuums" (ii) and (iii) do not represent themselves as genuine vacuum configurations, since they are defined only within a finite-volume space region below the de Sitter horizon. In order to obtain a well-defined static spherically symmetric configuration over the whole space–time, we need to match at the de Sitter horizon the "kinetic vacuums" (ii) and (iii) living in the interior de Sitter region to the exterior region with a nonstandard Reissner–Nordström–de Sitter geometry which carries an additional constant radial background electric field. Studying the vacuum energy densities inside and outside the de Sitter horizon shows that the inside energy density is higher than the outside one. Thus, the fully extended to the whole space–time "kinetic vacuums" (ii) and (iii) represent gravitational bag-like configurations where:

(a) The type (ii) gravitational bag mimics some of the properties of the MIT bag model[1,2] — finite volume space region with *deconfinement* and large energy density versus infinite volume exterior region with *confinement* and low energy density.
(b) Both type (ii) and type (iii) gravitational bags resemble some of the properties of the solitonic "constituent quark" model[37] — they are charged and carry "color" flux to infinity.

The plan of the paper is as follows. In Sec. 2, we describe in some detail the general formalism for the new class of gravity-matter systems defined in terms of two independent non-Riemannian integration measures. In Secs. 3 and 4, we describe the properties of the two flat regions of the Einstein-frame effective scalar potential and derive the static spherically symmetric vacuum configurations. In Sec. 5, we construct static spherically symmetric solutions representing gravitational bag-like configurations. We conclude in Sec. 6 with some discussions. In App. A, we briefly outline the canonical Hamiltonian treatment of the modified gravity-matter models with two non-Riemannian space–time volume-forms, which elucidates the physical meaning of the auxiliary fields defining the non-Riemannian volume-forms. In App. B following Ref. 28, we show that the presence of the "square-root" Maxwell term $\sqrt{-F^2}$ generates confining effective potential between quantized charged fermions in static spherically symmetric space–times.

2. Gravity-Matter System Coupled to Charge-Confining Nonlinear Gauge Field — A Formalism with Two Independent non-Riemannian Volume-Forms

We shall consider the following nonstandard gravity/nonlinear-gauge-field/matter system with an action of the general form involving two independent non-Riemannian integration measure densities generalizing the models studied in Refs. 25 and 24 (for simplicity we will use units where the Newton constant is taken as $G_{\rm Newton} = 1/16\pi$):

$$S = \int d^4x\, \Phi_1(A)\big[R + L^{(1)}\big] + \int d^4x\, \Phi_2(B)\left[L^{(2)} + \epsilon R^2 + \frac{\Phi(H)}{\sqrt{-g}}\right]. \qquad (1)$$

Here the following notations are used:

- $\Phi_1(A)$ and $\Phi_2(B)$ are two independent non-Riemannian volume-forms, i.e. generally covariant integration measure densities on the underlying space–time manifold:
$$\Phi_1(A) = \frac{1}{3!}\varepsilon^{\mu\nu\kappa\lambda}\partial_\mu A_{\nu\kappa\lambda}\,, \quad \Phi_2(B) = \frac{1}{3!}\varepsilon^{\mu\nu\kappa\lambda}\partial_\mu B_{\nu\kappa\lambda}\,, \tag{2}$$
defined in terms of field-strengths of two auxiliary 3-index antisymmetric tensor gauge fields. $\Phi_{1,2}$ take over the role of the standard Riemannian integration measure density $\sqrt{-g} \equiv \sqrt{-\det\|g_{\mu\nu}\|}$ in terms of the space–time metric $g_{\mu\nu}$.
- $R = g^{\mu\nu}R_{\mu\nu}(\Gamma)$ and $R_{\mu\nu}(\Gamma)$ are the scalar curvature and the Ricci tensor in the first-order (Palatini) formalism, where the affine connection $\Gamma^\mu_{\nu\lambda}$ is *a priori* independent of the metric $g_{\mu\nu}$. Note that in the second action term we have added a R^2 gravity term (again in the Palatini form). Let us recall that $R+R^2$ gravity within the second-order formalism (which was also the first inflationary model) was originally proposed in Ref. 38.
- $L^{(1,2)}$ denote two different Lagrangians of a single scalar matter field ("dilaton") and of an Abelian gauge field potential A_μ of the form:
$$L^{(1)} = -\frac{1}{2}g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi - V(\varphi) - \frac{f_0}{2}\sqrt{-F^2(g)}\,, \qquad V(\varphi) = f_1\exp\{-\alpha\varphi\}\,, \tag{3}$$
$$L^{(2)} = -\frac{b}{2}e^{-\alpha\varphi}g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi + U(\varphi) - \frac{1}{4e^2}F^2(g)\,, \quad U(\varphi) = f_2\exp\{-2\alpha\varphi\}\,, \tag{4}$$
where
$$F^2(g) = F_{\mu\nu}F_{\kappa\lambda}g^{\mu\kappa}g^{\nu\lambda}\,, \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu\,. \tag{5}$$
Here, α, f_1, f_2 are dimensionful positive parameter, whereas b is a dimensionless one. The choice of the scalar potentials in (3)–(4) is similar to the choice in Ref. 39.
- $\Phi(H)$ indicate the dual field strength of a third auxiliary 3-index antisymmetric tensor gauge field:
$$\Phi(H) = \frac{1}{3!}\varepsilon^{\mu\nu\kappa\lambda}\partial_\mu H_{\nu\kappa\lambda}\,, \tag{6}$$
whose presence is crucial for nontriviality of the model.

Concerning the explicit form of the non-Riemannian integration measure densities (2) let us note that any of the pertinent auxiliary 3-index antisymmetric tensor gauge fields, for instance, $A_{\mu\nu\lambda}$ can be in particular parametrized in terms of four auxiliary scalar fields $\{\phi^I\}_{I=1,\ldots,4}$:
$$A_{\mu\nu\lambda} = \frac{1}{4}\varepsilon_{IJKL}\phi^I\partial_\mu\phi^J\partial_\nu\phi^K\partial_\lambda\phi^L\,, \tag{7}$$
so that
$$\Phi_1(A) = \frac{1}{4!}\varepsilon^{\mu\nu\kappa\lambda}\varepsilon_{IJKL}\partial_\mu\phi^I\partial_\nu\phi^J\partial_\kappa\phi^K\partial_\lambda\phi^L = \det\left\|\frac{\partial\phi^I}{\partial x^\mu}\right\|\,, \tag{8}$$

acquires the form of a Jacobian. In a recent study[40] of general relativity as an extended canonical gauge theory a similar Jacobian representation of the covariant integration measure has appeared in terms of additional scalar fields. However, unlike the present case in the construction of Ref. 40 the additional scalar fields enter also in the proper Lagrangian.

In what follows we will stick to the representation (2).

The scalar field potentials and the separate locations of the standard Maxwell and the square-root Maxwell gauge field terms have been chosen in such a way that the original action (1) is invariant under global Weyl-scale transformations:

$$g_{\mu\nu} \to \lambda g_{\mu\nu}\,, \quad \Gamma^{\mu}_{\nu\lambda} \to \Gamma^{\mu}_{\nu\lambda}\,, \quad \varphi \to \varphi + \frac{1}{\alpha}\ln\lambda\,, A_{\mu} \to A_{\mu}\,,$$
$$A_{\mu\nu\kappa} \to \lambda A_{\mu\nu\kappa}\,, \quad B_{\mu\nu\kappa} \to \lambda^2 B_{\mu\nu\kappa}\,, \quad H_{\mu\nu\kappa} \to H_{\mu\nu\kappa}\,. \tag{9}$$

The equations of motion resulting from the action (1) are as follows. Variation of (1) with respect to affine connection $\Gamma^{\mu}_{\nu\lambda}$:

$$\int d^4x\,\sqrt{-g}\,g^{\mu\nu}\left(\frac{\Phi_1}{\sqrt{-g}} + 2\epsilon\,\frac{\Phi_2}{\sqrt{-g}}\,R\right)\left(\nabla_\kappa \delta\Gamma^{\kappa}_{\mu\nu} - \nabla_\mu \delta\Gamma^{\kappa}_{\kappa\nu}\right) = 0\,, \tag{10}$$

gives, following the analogous derivation in the Ref. 39, that $\Gamma^{\mu}_{\nu\lambda}$ becomes a Levi-Civita connection:

$$\Gamma^{\mu}_{\nu\lambda} = \Gamma^{\mu}_{\nu\lambda}(\bar{g}) = \frac{1}{2}\bar{g}^{\mu\kappa}\left(\partial_\nu \bar{g}_{\lambda\kappa} + \partial_\lambda \bar{g}_{\nu\kappa} - \partial_\kappa \bar{g}_{\nu\lambda}\right)\,, \tag{11}$$

with respect to to the Weyl-rescaled metric $\bar{g}_{\mu\nu}$:

$$\bar{g}_{\mu\nu} = (\chi_1 + 2\epsilon\chi_2 R) g_{\mu\nu}\,, \quad \chi_1 \equiv \frac{\Phi_1(A)}{\sqrt{-g}}\,, \quad \chi_2 \equiv \frac{\Phi_2(B)}{\sqrt{-g}}\,. \tag{12}$$

Variation of the action (1) with respect to auxiliary tensor gauge fields $A_{\mu\nu\lambda}$, $B_{\mu\nu\lambda}$ and $H_{\mu\nu\lambda}$ yields the equations:

$$\partial_\mu\left[R + L^{(1)}\right] = 0\,,$$
$$\partial_\mu\left[L^{(2)} + \epsilon R^2 + \frac{\Phi(H)}{\sqrt{-g}}\right] = 0\,, \tag{13}$$
$$\partial_\mu\left(\frac{\Phi_2(B)}{\sqrt{-g}}\right) = 0\,,$$

whose solutions read:

$$\frac{\Phi_2(B)}{\sqrt{-g}} \equiv \chi_2 = \mathrm{const}\,,$$
$$R + L^{(1)} = -M_1 = \mathrm{const}\,, \tag{14}$$
$$L^{(2)} + \epsilon R^2 + \frac{\Phi(H)}{\sqrt{-g}} = -M_2 = \mathrm{const}\,.$$

Here, M_1 and M_2 are arbitrary dimensionful and χ_2 arbitrary dimensionless integration constants. The appearance of M_1, M_2 signifies *dynamical spontaneous breakdown* of global Weyl-scale invariance under (9) due to the scale noninvariant solutions (second and third ones) in (14).

It is also very instructive to elucidate the physical meaning of the three arbitrary integration constants M_1, M_2, χ_2 from the point of view of the canonical Hamiltonian formalism. Namely, as shown in App. A M_1, M_2, χ_2 are identified as conserved Dirac-constrained canonical momenta conjugated to (certain components of) the auxiliary maximal rank antisymmetric tensor gauge fields $A_{\mu\nu\lambda}$, $B_{\mu\nu\lambda}$, $H_{\mu\nu\lambda}$ entering the original non-Riemannian volume-form action (1).

Varying (1) with respect to $g_{\mu\nu}$ and using relations (14) we have:

$$\chi_1\left[R_{\mu\nu}+\frac{1}{2}\Big(g_{\mu\nu}L^{(1)}-T^{(1)}_{\mu\nu}\Big)\right] \\ -\frac{1}{2}\chi_2\Big[T^{(2)}_{\mu\nu}+g_{\mu\nu}\big(\epsilon R^2+M_2\big)\,\text{-4e}RR_{\mu\nu}\Big]=0\,, \tag{15}$$

where χ_1 and χ_2 are defined in (12), and $T^{(1,2)}_{\mu\nu}$ are the energy–momentum tensors of the scalar + gauge field Lagrangians with the standard definitions:

$$T^{(1,2)}_{\mu\nu}=g_{\mu\nu}L^{(1,2)}-2\frac{\partial}{\partial g^{\mu\nu}}L^{(1,2)}\,. \tag{16}$$

Taking the trace of Eq. (15) and using again second relation (14) we solve for the scale factor χ_1:

$$\chi_1=2\chi_2\frac{T^{(2)}/4+M_2}{L^{(1)}-T^{(1)}/2-M_1}\,, \tag{17}$$

where $T^{(1,2)}=g^{\mu\nu}T^{(1,2)}_{\mu\nu}$.

Using second relation (14), Eq. (15) can be put in the Einstein-like form:

$$R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R=\frac{1}{2}g_{\mu\nu}\big(L^{(1)}+M_1\big)+\frac{1}{2\Omega}\big(T^{(1)}_{\mu\nu}-g_{\mu\nu}L^{(1)}\big) \\ +\frac{\chi_2}{2\chi_1\Omega}\Big[T^{(2)}_{\mu\nu}+g_{\mu\nu}\big(M_2+\epsilon(L^{(1)}+M_1)^2\big)\Big]\,, \tag{18}$$

where

$$\Omega=1-\frac{\chi_2}{\chi_1}2\epsilon\big(L^{(1)}+M_1\big)\,. \tag{19}$$

Let us note that (12), upon taking into account second relation (14) and (19), can be written as:

$$\bar{g}_{\mu\nu}=\chi_1\Omega\,g_{\mu\nu}\,. \tag{20}$$

Now, we can bring Eq. (18) into the standard form of Einstein equations for the rescaled metric $\bar{g}_{\mu\nu}$ (20), i.e. the Einstein-frame equations:

$$R_{\mu\nu}(\bar{g}) - \frac{1}{2}\bar{g}_{\mu\nu}R(\bar{g}) = \frac{1}{2}T^{\mathrm{eff}}_{\mu\nu}\,, \tag{21}$$

with energy–momentum tensor corresponding according to the definition (16):

$$T^{\mathrm{eff}}_{\mu\nu} = g_{\mu\nu}L_{\mathrm{eff}} - 2\frac{\partial}{\partial g^{\mu\nu}}L_{\mathrm{eff}}\,, \tag{22}$$

to the following effective (Einstein-frame) scalar field Lagrangian:

$$L_{\mathrm{eff}} = \frac{1}{\chi_1\Omega}\left\{L^{(1)} + M_1 + \frac{\chi_2}{\chi_1\Omega}\left[L^{(2)} + M_1 + \epsilon(L^{(1)} + M_1)^2\right]\right\}. \tag{23}$$

In order to explicitly write L_{eff} in terms of the Einstein-frame metric $\bar{g}_{\mu\nu}$ (20) we use the short-hand notation for the scalar kinetic term:

$$X \equiv -\frac{1}{2}\bar{g}^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi \tag{24}$$

and represent $L^{(1,2)}$ in the form:

$$\begin{aligned} L^{(1)} &= \chi_1\Omega X - V - \chi_1\Omega\,\frac{f_0}{2}\sqrt{-F^2(\bar{g})}\,,\\ L^{(2)} &= \chi_1\Omega\, b e^{-\alpha\varphi}X + U - (\chi_1\Omega)^2\frac{1}{4e^2}F^2(\bar{g})\,, \end{aligned} \tag{25}$$

with V and U as in (3)–(4).

From Eqs. (17) and (19), taking into account (25), we find:

$$\begin{aligned} \frac{1}{\chi_1\Omega} &= \frac{(V - M_1)}{2\chi_2[U + M_2 + \epsilon(V - M_1)^2]}\\ &\quad\times\left[1 - \chi_2\left(\frac{be^{-\alpha\varphi}}{V - M_1} - 2\epsilon\right)X - \epsilon\chi_2 f_0\sqrt{-F^2(\bar{g})}\right]. \end{aligned} \tag{26}$$

Upon substituting expression (26) into (23) we arrive at the explicit form for the Einstein-frame matter Lagrangian:

$$\begin{aligned} L_{\mathrm{eff}} &= A(\varphi)X + B(\varphi)X^2 - U_{\mathrm{eff}}(\varphi) - \frac{F^2(\bar{g})}{4e^2_{\mathrm{eff}}(\varphi)}\\ &\quad - \frac{f_{\mathrm{eff}}(\varphi)}{2}\sqrt{-F^2(\bar{g})} - \epsilon\chi_2 f_0 A(\varphi)X\sqrt{-F^2(\bar{g})}\,. \end{aligned} \tag{27}$$

The coefficient functions in (27) read:

$$\begin{aligned} A(\varphi) &= 1 - 4U_{\mathrm{eff}}(\varphi)\left[\epsilon\chi_2 - \frac{\chi_2 be^{-\alpha\varphi}}{2(V(\varphi) - M_1)}\right],\\ B(\varphi) &= \epsilon\chi_2 - 4U_{\mathrm{eff}}(\varphi)\left[\epsilon\chi_2 - \frac{\chi_2 be^{-\alpha\varphi}}{2(V(\varphi) - M_1)}\right]^2, \end{aligned} \tag{28}$$

whereas the effective scalar field potential reads:

$$U_{\rm eff}(\varphi) \equiv \frac{(V-M_1)^2}{4\chi_2[U+M_2+\epsilon(V-M_1)^2]} \\ = \frac{(f_1 e^{-\alpha\varphi}-M_1)^2}{4\chi_2\big[f_2 e^{-2\alpha\varphi}+M_2+\epsilon(f_1 e^{-\alpha\varphi}-M_1)^2\big]}\,, \tag{29}$$

where the explicit form of V and U (3)–(4) are inserted. Further, the original gauge coupling constants are here replaced by φ-dependent effective coupling constants:

$$f_{\rm eff}(\varphi) = f_0\big(1-4\epsilon\chi_2 U_{\rm eff}(\varphi)\big) \\ = f_0\frac{f_2 e^{-2\alpha\varphi}+M_2}{f_2 e^{-2\alpha\varphi}+M_2+\epsilon(f_1 e^{-\alpha\varphi}-M_1)^2}\,, \tag{30}$$

$$\frac{1}{e^2_{\rm eff}(\varphi)} = \chi_2\left[\frac{1}{e^2}+\epsilon f_0^2\big(1-4\epsilon\chi_2 U_{\rm eff}(\varphi)\big)\right] \\ = \chi_2\left[\frac{1}{e^2}+\epsilon f_0^2\frac{f_2 e^{-2\alpha\varphi}+M_2}{f_2 e^{-2\alpha\varphi}+M_2+\epsilon(f_1 e^{-\alpha\varphi}-M_1)^2}\right]. \tag{31}$$

Let us recall that the dimensionless integration constant χ_2 systematically appearing in most relations is the ratio of the original second non-Riemannian integration measure to the standard Riemannian one (12).

We observe that even if we start with *no* standard Maxwell kinetic term for the gauge field, i.e. taking the limit $e^2 \to \infty$ in the original action (1)–(4), we nevertheless obtain a *dynamically induced* Maxwell term in the Einstein-frame action (27) with effective running charge according to (31):

$$\frac{1}{e^2_{\rm eff}(\varphi)} = \epsilon\chi_2 f_0^2\frac{(f_2 e^{-2\alpha\varphi}+M_2)}{\big[f_2 e^{-2\alpha\varphi}+M_2+\epsilon(f_1 e^{-\alpha\varphi}-M_1)^2\big]}\,. \tag{32}$$

From (32) we see that dynamical Maxwell term generation is a cumulative effect of the simultaneous presence of the "confining" gauge field term $\sqrt{-F^2}$ and the R^2 gravity term.

3. Flat Regions of the Effective Scalar Potential and Nontrivial "Vacuum" Solutions

The explicit expressions for the effective potential $U_{\rm eff}(\varphi)$ (29), the scalar ("k-essence") kinetic terms' coefficient functions $A(\varphi)$ and $B(\varphi)$ (28) and the effective gauge coupling constants (30)–(31) reveal the following crucial feature of the Einstein-frame matter Lagrangian (27)–(31): the presence of *two infinitely large flat regions* — one for large negative and one for large positive values of the scalar field φ, where all of the above are essentially constant with respect to φ.

Depending on the sign of the integration constant M_1 we obtain two types of shapes for the effective scalar potential $U_{\rm eff}(\varphi)$ (29) depicted on Figs. 1 and 2.

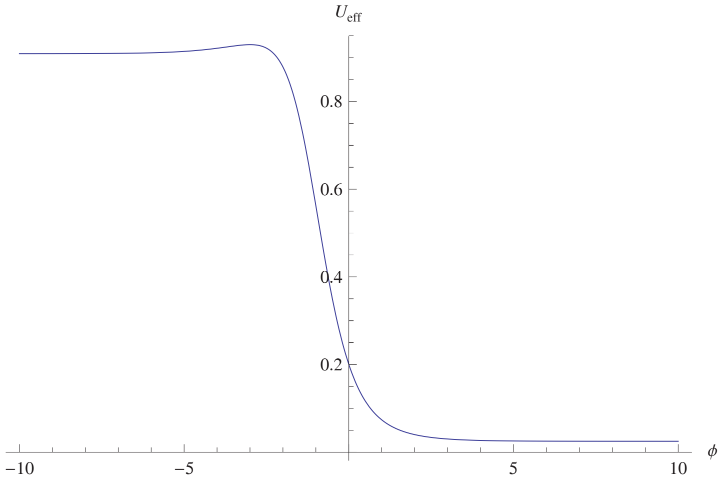

Fig. 1. Qualitative shape of the effective scalar potential $U_{\text{eff}}(\varphi)$ (29) for $M_1 > 0$.

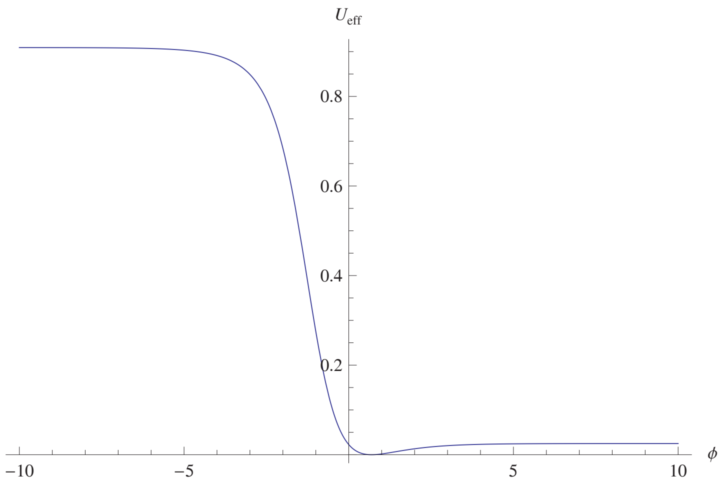

Fig. 2. Qualitative shape of the effective scalar potential $U_{\text{eff}}(\varphi)$ (29) for $M_1 < 0$.

For large negative values of φ we have for the effective potential and the coefficient functions in the Einstein-frame matter Lagrangian (27)–(31):

$$U_{\text{eff}}(\varphi) \simeq U_{(-)} \equiv \frac{f_1^2/f_2}{4\chi_2(1+\epsilon f_1^2/f_2)}\,, \tag{33}$$

$$\begin{aligned} A(\varphi) &\simeq A_{(-)} \equiv \frac{1+\frac{1}{2}bf_1/f_2}{1+\epsilon f_1^2/f_2}\,, \\ B(\varphi) &\simeq B_{(-)} \equiv \epsilon\chi_2\frac{1+bf_1/f_2-b^2/4\epsilon f_2}{1+\epsilon f_1^2/f_2}\,, \end{aligned} \tag{34}$$

$$e^2_{\rm eff}(\varphi) \simeq e_{(-)} \equiv \frac{e^2}{\chi_2}\frac{1+\epsilon f_1^2/f_2}{1+\epsilon f_1^2/f_2 + e^2\epsilon f_0^2}\,, \qquad f_{\rm eff}(\varphi) \simeq f_{(-)} \equiv \frac{f_0}{1+\epsilon f_1^2/f_2}\,. \tag{35}$$

This will be called "(−) flat region." In the second flat region for large positive φ, which will be called "(+) flat region" we have:

$$U_{\rm eff}(\varphi) \simeq U_{(+)} \equiv \frac{M_1^2/M_2}{4\chi_2(1+\epsilon M_1^2/M_2)}\,, \tag{36}$$

$$A(\varphi) \simeq A_{(+)} \equiv \frac{M_2}{M_2+\epsilon M_1^2}\,, \qquad B(\varphi) \simeq B_{(+)} \equiv \epsilon\chi_2\frac{M_2}{M_2+\epsilon M_1^2}\,, \tag{37}$$

$$e^2_{\rm eff}(\varphi) \simeq e_{(+)} \equiv \frac{e^2}{\chi_2}\frac{1+\epsilon M_1^2/M_2}{1+\epsilon M_1^2/M_2 + e^2\epsilon f_0^2}\,, \qquad f_{\rm eff}(\varphi) \simeq f_{(+)} \equiv \frac{f_0}{1+\epsilon M_1^2/M_2}\,. \tag{38}$$

The scalar and gauge field equations of motion resulting from the Einstein-frame Lagrangian (here $L_{\rm eff}$ is considered function of φ, X, F^2):

$$\frac{1}{\sqrt{-\bar{g}}}\partial_\mu\left(\sqrt{-\bar{g}}\bar{g}^{\mu\nu}\partial_\nu\varphi\frac{\partial L_{\rm eff}}{\partial X}\right) - \frac{\partial L_{\rm eff}}{\partial\varphi} = 0\,, \qquad \partial_\nu\left(\sqrt{-\bar{g}}F^{\mu\nu}\frac{\partial L_{\rm eff}}{\partial F^2}\right) = 0 \tag{39}$$

thanks to the presence of the two (±) large flat regions (33)–(35) and (36)–(38), as well as due to the "k-essence"-type nonlinear dependence of $L_{\rm eff}$ on the scalar kinetic term, allow for the following two classes of nontrivial "vacuum" solutions:

(i) "Standard vacuum" containing standard constant "dilaton" vacuum plus nontrivial gauge field vacuum:

$$\varphi = {\rm const} \to X = 0\,, \quad \frac{\partial L_{\rm eff}}{\partial\varphi} = 0\,, \quad \frac{\partial L_{\rm eff}}{\partial F^2} = 0\,. \tag{40}$$

Here, the value $\varphi = {\rm const}$ belongs to either the (−) flat region (33) or the (+) flat region (36) of the effective scalar potential.

(ii) "Kinetic vacuum" (this type of "vacuum" exists thanks to the nonlinear with respect to X "k-essence" nature of the effective Lagrangian (27)):

$$\frac{\partial L_{\rm eff}}{\partial X} = 0\,, \quad \frac{\partial L_{\rm eff}}{\partial\varphi} = 0\,, \quad \frac{\partial L_{\rm eff}}{\partial F^2} = 0\,. \tag{41}$$

In the first class of "standard vacuum" solutions the last Eq. (40) yields the following nontrivial "vacuum" value for the gauge field:

$$\sqrt{-F^2_{\rm vac}} = e^2_{\rm eff}(\varphi) f_{\rm eff}(\varphi)\,, \tag{42}$$

and for the associated matter energy–momentum tensor (cf. (22)) we get:

$$T^{\rm eff}_{\mu\nu} = \bar{g}_{\mu\nu} L_{\rm eff}\Big|_{X=0,\,\frac{\partial L_{\rm eff}}{\partial F^2}=0} = -\bar{g}_{\mu\nu} U^{(\rm standard)}_{\rm total}\,, \tag{43}$$

where $U^{(\rm standard)}_{\rm total}$ is the total effective scalar potential in the "standard vacuum" (40):

$$U^{(\rm standard)}_{\rm total} = U_{\rm eff} + \frac{1}{4} e^2_{\rm eff} f^2_{\rm eff} = U_{\rm eff} + \frac{e^2 f_0^2 \big(1 - 4\epsilon\chi_2 U_{\rm eff}\big)^2}{4\chi_2 \big[1 + e^2 \epsilon f_0^2 \big(1 - 4\epsilon\chi_2 U_{\rm eff}\big)\big]}\,. \tag{44}$$

In both ($\mp$) flat regions (33)–(35) and (36)–(38) we have correspondingly:

$$\sqrt{-F^2_{\rm vac}} \simeq \sqrt{-F^2_{(-)}} = \frac{e^2 f_0}{\chi_2\,(1 + \epsilon f_1^2/f_2 + e^2\epsilon f_0^2)}\,, \tag{45}$$

$$\sqrt{-F^2_{\rm vac}} \simeq \sqrt{-F^2_{(+)}} = \frac{e^2 f_0}{\chi_2\,(1 + \epsilon M_1^2/M_2 + e^2\epsilon f_0^2)}\,, \tag{46}$$

and

$$U^{(\rm standard)}_{\rm total} \simeq U^{(\rm standard)}_{(-)} \equiv \frac{1}{4\epsilon\chi_2}\left[1 - \frac{1}{1 + \epsilon f_1^2/f_2 + \epsilon e^2 f_0^2}\right], \tag{47}$$

$$U^{(\rm standard)}_{\rm total} \simeq U^{(\rm standard)}_{(+)} \equiv \frac{1}{4\epsilon\chi_2}\left[1 - \frac{1}{1 + \epsilon M_1^2/M_2 + \epsilon e^2 f_0^2}\right]. \tag{48}$$

Therefore, according to (43) the solutions of the Einstein-frame $\bar{g}_{\mu\nu}$-equations are of de Sitter-type (pure de Sitter or Schwarzschild–de Sitter):

$$ds^2 = \bar{g}_{\mu\nu}\, dx^\mu\, dx^\nu = -\mathcal{A}(r) dt^2 + \frac{dr^2}{\mathcal{A}(r)} + r^2 (d\theta^2 + \sin^2\theta\, d\phi)\,, \tag{49}$$

$$\mathcal{A}(r) = 1 - \frac{\Lambda_{(\pm)}}{3} r^2\,, \quad \text{or} \quad \mathcal{A}(r) = 1 - \frac{2m}{r} - \frac{\Lambda_{(\pm)}}{3} r^2\,, \tag{50}$$

in static spherically symmetric coordinate chart, with effective cosmological constants $\Lambda_{(\pm)}$ given by (47)–(48):

$$\Lambda_{(-)} \equiv \Lambda^{(\rm standard)}_{(-)} = \frac{1}{2} U^{(\rm standard)}_{(-)} \quad \text{in the } (-) \text{ flat region } (33)\text{–}(35)\,, \tag{51}$$

$$\Lambda_{(+)} \equiv \Lambda^{(\rm standard)}_{(+)} = \frac{1}{2} U^{(\rm standard)}_{(+)} \quad \text{in the } (+) \text{ flat region } (36)\text{–}(38)\,. \tag{52}$$

From the above analysis of the "standard vacuum" solutions — the one corresponding to $\varphi = \text{const}$ belonging to the ($-$) flat region of the effective scalar potential with nonzero gauge field vacuum value and vacuum energy density as in (45), (47), and the second, the one corresponding to $\varphi = \text{const}$ belonging to the (+) flat region of the effective scalar potential with gauge field vacuum value and vacuum energy density as in (46), (48) — we conclude that these "standard vacuum" solutions describe *charge confining* phases with dynamically generated cosmological constants (51) and (52).

Indeed, according to 't Hooft's confinement proposal,[32,33] and as shown explicitly in Ref. 28 in the case of flat space–time, the nonzero vacuum values of the gauge field (45), (46) imply *confinement* dynamics of charged particles with the strength of confinement proportional to these same gauge field vacuum values. Namely, under plausible truncation for static spherically symmetric configurations the canonically quantized theory in flat space–time of charged fermions interacting with the nonlinear gauge fields with the "square-root" Maxwell term produces an effective "Cornell"-type potential $V_{\rm eff} = -\frac{\alpha}{L} + \beta L$ (see also Eq. (88)) between quantized fermions separated by a distance L and where β is proportional to the coupling constant of the "square-root" Maxwell term, which in turn is proportional to the nonzero vacuum value of the gauge field.

The formalism to prove confinement used in Ref. 28 can be easily generalized to the case of curved static spherically symmetric space–times, in particular for de Sitter space–time where both charged fermions are located within the interior de Sitter region below the de Sitter horizon ($r \leq \sqrt{3/\Lambda_{(\pm)}}$) — see App. B.

4. "Kinetic Vacuum" Solutions

We now turn our attention to the second class of "kinetic vacuum" solutions (41). The equations $\frac{\partial L_{\rm eff}}{\partial X} = 0$ and $\frac{\partial L_{\rm eff}}{\partial F^2} = 0$ yield:

$$X_{\rm kin} = -\frac{A}{2B}\,\frac{1-\epsilon\chi_2 f_0 f_{\rm eff} e^2_{\rm eff}}{1-\epsilon^2\chi_2^2 f_0^2 e^2_{\rm eff} A^2/B}\,, \tag{53}$$

$$\sqrt{-F^2_{\rm kin}} = e^2_{\rm eff}\frac{f_{\rm eff} - \epsilon\chi_2 f_0 A^2/B}{1-\epsilon^2\chi_2^2 f_0^2 e^2_{\rm eff} A^2/B}\,. \tag{54}$$

Using the identity $\frac{\partial L_{\rm eff}}{\partial X} = 2BX + A(1-\epsilon\chi_2 f_0\sqrt{-F^2})$ we can rewrite the Einstein-frame Lagrangian $L_{\rm eff}$ (23) in the form:

$$L_{\rm eff} = \frac{1}{4B}\left(\frac{\partial L_{\rm eff}}{\partial X}\right)^2 - \tilde{U}(\varphi, F^2)\,, \tag{55}$$

with

$$\tilde{U}(\varphi, F^2) = U_{\rm eff} + \frac{A^2}{4B} - \frac{1}{2}\sqrt{-F^2}\left(f_{\rm eff} - \epsilon\chi_2 f_0\frac{A^2}{B}\right) - \frac{1}{4}F^2\left(\frac{1}{e^2_{\rm eff}} - \epsilon^2\chi_2^2 f_0^2\frac{A^2}{B}\right). \tag{56}$$

Inserting in (55)–(56) the "on-shell" values (53)–(54), we obtain for the matter energy–momentum tensor:

$$T^{\rm eff}_{\mu\nu} = \bar{g}_{\mu\nu} L_{\rm eff}\Big|_{\frac{\partial L_{\rm eff}}{\partial X}=0,\,\frac{\partial L_{\rm eff}}{\partial F^2}=0} = -\bar{g}_{\mu\nu} U^{(\rm kinetic)}_{\rm total}\,, \tag{57}$$

where $U^{(\text{kinetic})}_{\text{total}}$ is the total effective scalar potential in the "kinetic vacuum" (41):

$$U^{(\text{kinetic})}_{\text{total}} = U_{\text{eff}} + \frac{A^2}{4B} + \frac{1}{4}e^2_{\text{eff}}\frac{\left(f_{\text{eff}} - \epsilon\chi_2 f_0 \frac{A^2}{B}\right)^2}{1 - e^2_{\text{eff}}\epsilon^2\chi_2^2 f_0^2 \frac{A^2}{B}} . \tag{58}$$

Note from Eqs. (55)–(56) that within the "kinetic vacuum" $\frac{\partial L_{\text{eff}}}{\partial X} = 0$ the effective gauge coupling constants become:

$$\tilde{f}_{\text{eff}} = f_{\text{eff}} - \epsilon\chi_2 f_0 \frac{A^2}{B} , \quad \tilde{e}^2_{\text{eff}} = \frac{e^2_{\text{eff}}}{1 - e^2_{\text{eff}}\epsilon^2\chi_2^2 f_0^2 \frac{A^2}{B}} . \tag{59}$$

4.1. *"Kinetic vacuum" in the (+) flat region of effective scalar potential*

First, we consider the "kinetic vacuum" solution in the (+) flat region. Using (36)–(38) in (54), (53) and (58) we obtain from (59):

$$\tilde{f}_{\text{eff}} \equiv \tilde{f}_{(+)} = f_{(+)} - \epsilon\chi_2 f_0 \frac{A^2_{(+)}}{B_{(+)}} = 0 , \tag{60}$$

which yields:

$$\sqrt{-F^2_{\text{kin}}}\Big|_{(+)} = 0 , \quad X_{\text{kin}} \simeq X_{(+)} = -\frac{A_{(+)}}{2B_{(+)}} = -\frac{1}{2\epsilon\chi_2} , \tag{61}$$

$$U^{(\text{kinetic})}_{\text{total}} \simeq U^{(\text{kinetic})}_{(+)} = \frac{1}{4\epsilon\chi_2} \to T^{\text{eff}}_{\mu\nu} = -\bar{g}_{\mu\nu}\frac{1}{4\epsilon\chi_2} , \tag{62}$$

i.e. we have here an effective cosmological constant:

$$\Lambda_{(+)} \equiv \Lambda^{(\text{kinetic})}_{(+)} = \frac{1}{8\epsilon\chi_2} . \tag{63}$$

Let us particularly stress on the first relation in (61) — the zero vacuum value for the nonlinear gauge field, which is due to the vanishing (60) of the effective coupling constant of the "square-root" Maxwell term. Again, in accordance with 't Hooft's confinement proposal[32,33] and as demonstrated explicitly in Ref. 28 and in App. B the latter implies absence of confinement of charged particles, i.e. the "kinetic vacuum" (61)–(62) describes a *deconfinement* phase.

According to (57) and (62)–(63) the solutions of the Einstein-frame $\bar{g}_{\mu\nu}$-equations in the "kinetic vacuum" are again of de Sitter-type (49)–(50) with $\Lambda_{(+)}$ given by (63).

The equation for the "dilaton" "kinetic vacuum" (second Eq. (61)) reads explicitly:

$$\bar{g}^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi - \frac{1}{\epsilon\chi_2} = 0 . \tag{64}$$

It has precisely the form of Hamilton–Jacobi equation for the Hamilton–Jacobi action:

$$S(x) \equiv \varphi(x) = \frac{1}{\sqrt{\epsilon\chi_2}}\int_{\lambda_{\text{in}}}^{\lambda_{\text{out}}} d\lambda \sqrt{g_{\mu\nu}(x(\lambda))\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}} , \tag{65}$$

corresponding to spacelike geodesics $x^\mu(\lambda)$ starting from some fixed point $x_{(0)}$ (e.g. $x_{(0)} = 0$) at a fixed value of the affine parameter $\lambda_{\rm in}$ and passing through $x = x(\lambda_{\rm out})$ at $\lambda_{\rm out}$. This Hamilton–Jacobi action (65) measures the proper distance between the points $x_{(0)}$ and x on the manifold modulo the numerical factor $1/\sqrt{\epsilon\chi_2}$ (for modern pedagogical exposition, see, e.g. Ref. 41).

A static spherically symmetric solution for $\varphi(x)$ is given by:

$$\left(\frac{\partial\varphi}{\partial r}\right)^2 = \frac{1}{\epsilon\chi_2\,\mathcal{A}(r)} \to \varphi(r) = \varphi_{(+)} + \frac{1}{\sqrt{\epsilon\chi_2}}\int^r \frac{dr'}{\sqrt{\mathcal{A}(r')}}\,, \tag{66}$$

where the initial value $\varphi_{(+)}$ must belong to the $(+)$ flat region (36)–(38) (large positive φ).

In the case of pure de Sitter metric (49)–(50) the solution $\varphi(r)$ (66), measuring the proper radial distance between 0 and r, is clearly defined only for r in the interval $r \in \big(0, r_{(+)}\big)$, where

$$r_{(+)} = \sqrt{24\epsilon\chi_2}\,, \tag{67}$$

is the de Sitter horizon radius. The solution $\varphi(r)$ reads explicitly:

$$\varphi(r) = \varphi_{(+)} + \sqrt{24}\arcsin\left(\frac{r}{r_{(+)}}\right), \tag{68}$$

where the initial value $\varphi_{(+)}$ belongs to the $(+)$ flat region of the effective scalar potential. Let us also recall that the integral in the second Eq. (66), which is equal to $r_{(+)}\arcsin\big(\frac{r}{r_{(+)}}\big)$, yields the proper radial distance in the internal de Sitter region $r \leq r_{(+)}$.

In the case of Schwarzschild–de Sitter metric (49)–(50) the solution $\varphi(r)$ (66) is defined in the interval $r \in (r_S, r_H)$ between the inner (Schwarzschild-type) horizon r_S and the outer (de Sitter-type) horizon r_H. In what follows, we will consider the case of pure de Sitter metric.

Since the "kinetic vacuum" corresponding to the $(+)$ flat region described by (61)–(68) is defined only within the finite-volume space region below the de Sitter horizon, in order to be extended to the whole space it must be matched to another spherically symmetric configuration with the standard constant "dilaton" vacuum defined in the outer region beyond the de Sitter horizon with

$$\varphi = \varphi(r_{(+)}) = \varphi_{(+)} + \sqrt{6}\pi = \text{const} \quad \text{for} \quad r > r_{(+)}\,, \tag{69}$$

where the latter is the limiting value of (68) at the horizon. The corresponding construction yields a gravitational bag-like solution mimicking both some of the features of the MIT bags in QCD phenomenology[1,2] as well as some of the features of the "constituent quark" model of Ref. 37 to be discussed in the next section.

4.2. *"Kinetic vacuum" in the $(-)$ flat region of effective scalar potential*

Next, let us consider the "kinetic vacuum" solution corresponding to the $(-)$ flat region. In this case using (33)–(35) in (54), (53) and (58) and introducing short-hand notations for some combinations of the parameters to simplify the resulting expressions:

$$\xi \equiv b\frac{f_1}{f_2}\,, \quad \gamma \equiv \frac{f_2}{4\epsilon f_1^2}\,, \quad \beta \equiv e^2\frac{f_0^2 f_2^2}{16\epsilon f_1^4} = \epsilon e^2 f_0^2 \gamma^2\,, \tag{70}$$

we obtain:

$$\sqrt{-F_{\text{kin}}^2} \simeq \sqrt{-F_{\text{kin}}^2}\Big|_{(-)} = \frac{\beta\xi^4}{\epsilon\chi_2(1+\xi-\xi^2\gamma)[1+\xi-\xi^2\gamma(1+\beta/\gamma^2)]}\,, \tag{71}$$

$$X_{\text{kin}} \simeq X_{(-)} = -\frac{1}{2\epsilon\chi_2}\,\frac{1+\xi/2}{1+\xi-\xi^2\gamma(1+\beta/\gamma^2)}\,, \tag{72}$$

$$U_{\text{total}}^{(\text{kinetic})} \simeq U_{(-)}^{(\text{kinetic})} = \frac{1}{4\epsilon\chi_2}\,\frac{1+\xi-\xi^2\beta/\gamma}{1+\xi-\xi^2\gamma(1+\beta/\gamma^2)}\,, \tag{73}$$

$$T_{\mu\nu}^{\text{eff}} = -\bar{g}_{\mu\nu}U_{(-)}^{(\text{kinetic})} \equiv -\bar{g}_{\mu\nu}2\Lambda_{(-)}\,. \tag{74}$$

The space–time metric is again of de Sitter-type (49)–(50) with effective cosmological constant:

$$\Lambda_{(-)} \equiv \Lambda_{(-)}^{(\text{kinetic})} = \frac{1}{8\epsilon\chi_2}\,\frac{1+\xi-\xi^2\beta/\gamma}{1+\xi-\xi^2\gamma(1+\beta/\gamma^2)}\,. \tag{75}$$

According to Eq. (71) the vacuum value of the gauge field is nonzero (except in the special case $\xi = 0$, see (83)), therefore, following 't Hooft's confinement proposal[32,33] and Refs. 27–31 we conclude that the "kinetic vacuum" (71)–(73) supports dynamics of charged particles as in the "standard vacuum" phase (40) except in the special case $\xi = 0$.

The qualitative shape of the "kinetic vacuum" energy density in the $(-)$ flat region $U_{(-)}^{(\text{kinetic})}(\xi)$ (73) as function of the parameter $\xi \equiv b\frac{f_1}{f_2}$ (70) is depicted in Fig. 3.

$U_{(-)}^{(\text{kinetic})}(\xi)$ has a local minimum at $\xi = 0$, i.e. at $b = 0$ where

$$U_{(-)}^{(\text{kinetic})}(0) = 1/4\epsilon\chi_2\,, \tag{76}$$

and it raises to $+\infty$ at $\xi = \xi_\gamma^{(\pm)}$ which are the roots of the quadratic expression $1+\xi-\xi^2\gamma(1+\beta/\gamma^2$:

$$\xi_\gamma^{(\pm)} = \frac{1}{2\gamma(1+\beta/\gamma^2)}\left[1 \pm \sqrt{1+4\gamma(1+\beta/\gamma^2)}\right]. \tag{77}$$

Alternatively, for large $|\xi|$, $U_{(-)}^{(\text{kinetic})}(\xi \to \pm\infty) = \frac{1}{4\epsilon\chi_2}\left(1+1/\epsilon e^2 f_0^2\right)^{-1}$.

The scalar field for static spherically symmetric configurations is given by:

$$\left(\frac{\partial\varphi}{\partial r}\right)^2 = -2\frac{X_{(-)}}{\mathcal{A}(r)}\,, \tag{78}$$

therefore, solutions exist only for the range of parameters for which $X_{(-)} < 0$. From the explicit expression (72) we find:

$$X_{(-)} < 0 \quad \text{for} \quad \xi_\gamma^{(-)} < \xi < \xi_\gamma^{(+)} \quad \text{and for} \quad \xi < -2\,, \tag{79}$$

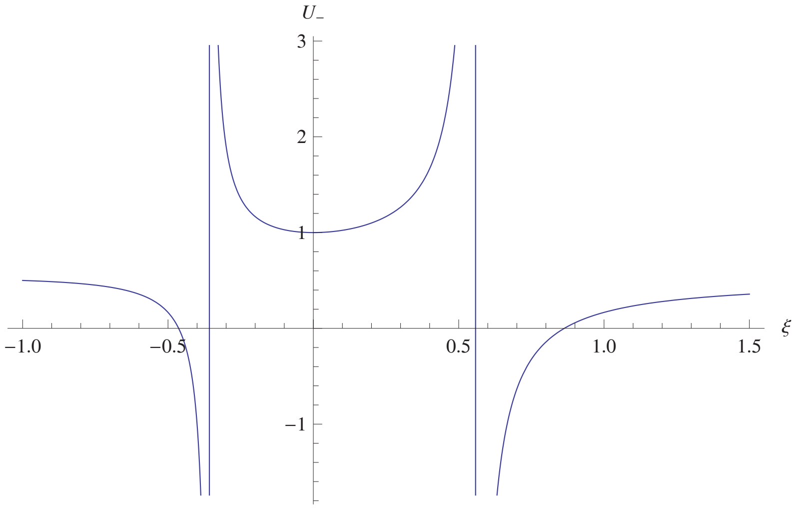

Fig. 3. Qualitative shape of the "kinetic vacuum" energy density (73) as function of $\xi \equiv b\frac{f_1}{f_2}$ (70).

where $\xi_\gamma^{(\pm)}$ are the same as in (77). Thus, the solution of (78) is similar to (68):

$$\varphi(r) = \varphi_{(-)} + \sqrt{2|X_{(-)}|}\int^r \frac{dr'}{\sqrt{\mathcal{A}(r')}} \\ = \varphi_{(-)} + \sqrt{2|X_{(-)}|}r_{(-)} \arcsin\left(\frac{r}{r_{(-)}}\right), \qquad (80)$$

where the initial value $\varphi_{(-)}$ must belong to the $(-)$ flat region (33)–(35) (large negative φ), $r_{(-)}$ is the de Sitter horizon radius:

$$r_{(-)} = \sqrt{\frac{3}{\Lambda_{(-)}}} \quad \left(\Lambda_{(-)} \text{ as in (75)}\right), \qquad (81)$$

$|X_{(-)}|$ is given by (72), and again $\varphi(r)$ is defined only in the space region inside the de Sitter horizon ($r \leq r_{(-)}$).

Thus, similarly to the previous case for the $(+)$ flat region, here in the $(-)$ flat region again we need to match the "kinetic vacuum" given by (71)–(80) to another spherically symmetric configuration with the standard constant dilaton vacuum:

$$\varphi = \varphi(r_{(-)}) = \varphi_{(-)} + \pi r_{(-)}\sqrt{\frac{|X_{(-)}|}{2}} = \text{const} \quad \text{for} \quad r > r_{(-)}\,, \qquad (82)$$

defined in the outer region, where $\varphi(r_{(-)})$ is the limiting value of $\varphi(r)$ (80) at the horizon. This will be considered in the next section.

Let us specifically note that in the special case $\xi \equiv b\frac{f_1}{f_2} = 0$, i.e. for $b = 0$ meaning that in this case the noncanonical scalar kinetic term is absent from the original second Lagrangian $L^{(2)}$ (4), the expressions (71)–(73) and (75), (81) drastically simplify:

$$\left(\sqrt{-F^2_{\rm kin}}\big|_{(-)}\right)\Big|_{\xi=0} = 0\,, \quad X_{(-)}|_{\xi=0} = -\frac{1}{2\epsilon\chi_2}\,, \quad U^{(\rm kinetic)}_{(-)}\Big|_{\xi=0} = \frac{1}{4\epsilon\chi_2}\,, \\ \Lambda^{(\rm kinetic)}_{(-)}\Big|_{\xi=0} = \frac{1}{8\epsilon\chi_2}\,, \quad r_{(-)}|_{\xi=0} = \sqrt{24\epsilon\chi_2}\,. \tag{83}$$

Expressions (83) precisely coincide with the corresponding values of $\sqrt{-F^2_{\rm kin}}\big|_{(+)} = 0$, $X_{(+)}$, $U^{(\rm kinetic)}_{(+)}$, $\Lambda^{(\rm kinetic)}_{(+)}$, $r_{(+)}$ (61)–(63), (67) in the (+) flat region of the effective scalar potential. In particular, from the first relation in (83) we conclude that in the special case $\xi = 0$ ($b = 0$) the corresponding "kinetic vacuum" on the (−) flat region of the effective scalar potential implies deconfinement in complete analogy with the "kinetic vacuum" on the (+) flat region (Subsec. 4.1).

5. Gravitational Bag-like Solutions

5.1. *Matching "kinetic vacuum" to standard constant "dilaton" vacuum in (+) flat region*

First, we construct matching of the "kinetic vacuum" in (+) flat region of the effective scalar potential given by de Sitter metric (49)–(50) in the interior region $\left(r < r_{(+)}\right)$ below the de Sitter horizon $r_{(+)} = \sqrt{24\epsilon\chi_2}$ with effective cosmological constant (63) and by Eqs. (61)–(68), to a static spherically symmetric configuration containing the standard constant "dilaton" vacuum (69) in the outer region $\left(r > r_{(+)}\right)$ beyond the de Sitter horizon. The "matching" specifically means that the "dilaton" field, the gauge field strength and the metric with its first derivatives must be continuous across the horizon, in particular, the de Sitter horizon of the interior metric must coincide with a horizon of the exterior metric.

Obviously, the static spherically symmetric configuration in the outer region cannot be the "standard vacuum" of the full "dilaton" plus gauge field subsystem given by (40), (42) and (48), since:

(a) In the inner "kinetic vacuum" region $\sqrt{-F^2_{\rm kin}}\big|_{(+)} = 0$ (first Eq. (61)), whereas in the outer "standard vacuum" region $\sqrt{-F^2_{(+)}} \neq 0$ (46), which would imply that there should be a lightlike ("null") brane with a nonzero surface electric charge density located on de Sitter horizon to account for the jump of the gauge field strength across the horizon.

(b) The effective cosmological constant in the outer "standard vacuum" region (52) is different and smaller than the effective cosmological constant (63) in the inner "kinetic vacuum" region.

In Refs. 42 and 43 (see also the earlier works[34–36]) we have already explicitly derived static spherically symmetric solutions of the coupled gravity/nonlinear gauge field/scalar "dilaton" system (27) with a generalized Reissner–Nordström–(anti-)de Sitter geometry carrying a nonvanishing background constant radial electric field in addition to the standard Coulomb field. We will use this type of solution in the outer region beyond the de Sitter horizon to be matched with the "kinetic vacuum" (61)–(68) in the interior region.

Specifically, for $r > r_{(+)} = \sqrt{24\epsilon\chi_2}$ the solution reads[42,43] (the additional factor $\frac{1}{16\pi}$ in Eq. (85) is due to the adopted normalization for the Newton constant $G_{\rm Newton} = 1/16\pi$):

$$ds^2 = -\mathcal{A}_{\rm out}(r)dt^2 + \frac{dr^2}{\mathcal{A}_{\rm out}(r)} + r^2(d\theta^2 + \sin^2\theta\, d\phi)\,, \tag{84}$$

$$\mathcal{A}_{\rm out}(r) = 1 + \frac{1}{16\pi}\left[-\sqrt{8\pi}|Q|f_{(+)} - \frac{2m}{r} + \frac{Q^2}{e^2_{(+)}\,r^2}\right] - \frac{\Lambda_{\rm out}}{3}r^2\,, \tag{85}$$

$$\begin{aligned} \Lambda_{\rm out} &= \frac{1}{2}U^{(\rm standard)}_{(+)} = \frac{1}{8\epsilon\chi_2}\left[1 - \frac{1}{1+\epsilon M_1^2/M_2 + \epsilon e^2 f_0^2}\right] \\ &\quad (\text{as in (48), (52)})\,, \\ \sqrt{-F^2_{\rm out}}(r) &= \sqrt{2}|\mathbf{E}_{\rm out}(r)| = e^2_{(+)}f_{(+)} - \frac{|Q|}{\sqrt{2\pi}\,r^2} \\ &\quad \left(e^2_{(+)},\, f_{(+)} \text{ as in (38)}\right)\,, \end{aligned} \tag{86}$$

$$\varphi = \varphi(r_{(+)}) = \text{const} \quad (\text{as in (69)})\,. \tag{87}$$

Notice that in (86) we have taken the constant radial background electric field and the Coulomb field with opposite directions. Let us also point out that the scalar potential corresponding to the static radial electric field in (86):

$$E^r_{\rm out} = -F_{0r} = \partial_r A_0(r)\,, \quad A_0(r) = \frac{1}{\sqrt{2}}e^2_{(+)}f_{(+)}r + \frac{|Q|}{\sqrt{4\pi}\,r}\,, \tag{88}$$

which is a static spherically symmetric solution of the nonlinear gauge field equations (last Eq. (39) with $L_{\rm eff}$ as in (27) and with $X = 0$ — "standard dilaton vacuum"), resembles the form of the well-known phenomenological "Cornell potential" in QCD which contains both a linear confining and a standard Coulomb part.[44–46]

The mass and electric charge parameters (m, Q) in (85) are to be determined from the matching at the common horizon:

$$\mathcal{A}_{\rm out}\big(r_{(+)}\big) = \mathcal{A}_{\rm in}\big(r_{(+)}\big) = 0\,, \quad \partial_r\mathcal{A}_{\rm out}\big(r_{(+)}\big) = \partial_r\mathcal{A}_{\rm in}\big(r_{(+)}\big)\,, \tag{89}$$

$$\sqrt{-F^2_{\rm out}}\big(r_{(+)}\big) = \sqrt{-F^2_{\rm kin}}\Big|_{(+)} = 0 \quad (\text{according to first Eq. (61)})\,, \tag{90}$$

with the "kinetic vacuum" (61)–(68) for $r < r_{(+)} = \sqrt{24\epsilon\chi_2}$ with:

$$ds^2 = -\mathcal{A}_{\rm in}(r)dt^2 + \frac{dr^2}{\mathcal{A}_{\rm in}(r)} + r^2\big(d\theta^2 + \sin^2\theta\, d\phi\big)\,, \tag{91}$$

$$\mathcal{A}_{\rm in}(r) = 1 - \frac{\Lambda_{\rm in}}{3}r^2\,, \quad \Lambda_{\rm in} = \frac{1}{2}U^{\rm (kinetic)}_{(+)} = \frac{1}{8\epsilon\chi_2}\,. \tag{92}$$

Inserting (86) into (90) we determine Q:

$$|Q| = \sqrt{2\pi}e^2_{(+)}f_{(+)}24\epsilon\chi_2 = \frac{\sqrt{2\pi}24\epsilon e^2 f_0}{1 + \epsilon M_1^2/M_2 + e^2\epsilon f_0^2}\,. \tag{93}$$

Now, inserting (93) into (89) yields:

$$m = 0\,, \tag{94}$$

and the following relation between the integration constants $M_{1,2}$ and the initial coupling constants ϵ, e, f_0:

$$1 + \epsilon\frac{M_1^2}{M_2} - 3\epsilon f_0^2 e^2 = 0\,. \tag{95}$$

To recapitulate, we have obtained the following "vacuum-like" solution:

- In the inner space region $r < r_{(+)} = \sqrt{24\epsilon\chi_2}$ we have an interior de Sitter region (91)–(92) below the de Sitter horizon at $r = r_{(+)}$ with effective cosmological constant (92), with vanishing vacuum gauge field (first Eq. (61)), "kinetic vacuum" scalar "dilaton" according to (68) and vacuum energy density (62):

$$\rho_{\rm in} \simeq U^{\rm (kinetic)}_{(+)} = \frac{1}{4\epsilon\chi_2}\,. \tag{96}$$

- In the outer space region $r > r_{(+)} = \sqrt{24\epsilon\chi_2}$ we have static spherically symmetric metric (85) with:

$$\mathcal{A}_{\rm out}(r) = 1 - \frac{1}{2\epsilon e^2 f_0^2} + \frac{6\chi_2}{e^2 f_0^2}\,\frac{1}{r^2} - \frac{r^2}{24\epsilon\chi_2}\left(1 - \frac{1}{4\epsilon e^2 f_0^2}\right), \tag{97}$$

  where we have used the explicit expressions for $e^2_{(+)}$, $f_{(+)}$ (38), Q (93) and $U^{\rm (standard)}_{(+)}$ (48) together with the relation (95). The metric with (97) has de Sitter-type horizon again at $r = r_{(+)}$ where the relation (95) among the parameters holds.
- The outside nonlinear gauge field (86) is a static radial electric field of the explicit form:

$$\sqrt{-F^2_{\rm out}(r)} = \sqrt{2}\left|E^r_{\rm out}(r)\right| = \frac{1}{4\epsilon\chi_2 f_0}\left(1 - \frac{24\epsilon\chi_2}{r^2}\right), \tag{98}$$

  where again we have used (38) and (95). In (98) there is a Coulomb piece in addition to a nonzero background constant radial electric field:

$$\left|E^r_{\rm background}\right| = \frac{1}{\sqrt{2}}e^2_{(+)}f_{(+)} = \frac{1}{\sqrt{2}\,4\epsilon\chi_2 f_0}\,. \tag{99}$$

  Thanks to the latter, the Coulomb field is completely cancelled at the horizon.

- The scalar "dilaton" is constant (87) and the energy density ($\rho = -T^0_0$) reads (using again (38), (48) and (95)):

$$\rho_{\rm out}(r) \simeq U^{(\rm standard)}_{(+)} - e^2_{(+)} f^2_{(+)} \left(\frac{r^2_{(+)}}{2r^2} - \frac{r^4_{(+)}}{4r^4} \right)$$

$$= \frac{1}{4\epsilon\chi_2}\left(1 - \frac{1}{4\epsilon e^2 f_0^2}\right) - \frac{1}{\epsilon e^2 f_0^2 r^2}\left(1 - \frac{24\epsilon\chi_2}{r^2}\right). \tag{100}$$

Obviously (recall in (100) $r > r_{(+)} \equiv \sqrt{24\epsilon\chi_2}$):

$$\rho_{\rm out}(r) \leq U^{(\rm standard)}_{(+)} = \frac{1}{4\epsilon\chi_2}\left[1 - \frac{1}{1+\epsilon M_1^2/M_2 + \epsilon e^2 f_0^2}\right] < \rho_{\rm in} = \frac{1}{4\epsilon\chi_2}. \tag{101}$$

The above solution (84)–(101) is a gravitational bag-like configuration on the (+) flat region of the effective scalar potential which mimics some of the properties of the MIT bag.[1,2] Indeed, as already noticed in Sec. 3:

(i) In the inner finite volume space region below the horizon $\left(r < r_{(+)}\right)$ the vanishing vacuum value of the gauge field (first Eq. (61)) implies absence of confinement of charged particles.[27–31]

(ii) According to (101) the vacuum energy density $\rho_{\rm in}$ in the inner finite volume space region (for $r < r_{(+)}$) is larger than the energy density $\rho_{\rm out}$ in the outside region.

There are, however, other properties of the present gravitational "bag" solution which are substantially different from those of the MIT bag and which rather resemble some of the properties of the solitonic "constituent quark" model:[37]

  (a) It is charged (the overall charge Q is nonzero (93)).
  (b) It carries nonzero "color" flux to infinite — because of the nonzero background constant radial electric field (99).

5.2. *Matching "kinetic vacuum" to standard constant "dilaton" vacuum in (−) flat region*

Using the same procedure above we can construct the matching of the "kinetic vacuum" (71)–(80) in (−) flat region of the effective scalar potential given by de Sitter metric (49)–(50) in the corresponding interior region $\left(r < r_{(-)}\right)$ below the de Sitter horizon $r_{(-)}$ (81), with effective cosmological constant (75), to a static spherically symmetric configuration containing the standard constant "dilaton" vacuum (82) in the outer region $\left(r > r_{(-)}\right)$ beyond the de Sitter horizon.

In the special case $\xi = 0$ ($b = 0$), as already noted in (83) above, the expressions (71)–(73) and (75) in the (−) flat region precisely coincide with the corresponding expressions (61)–(63), (67) in the (+) flat region. Taking into account (83) and the explicit form of $e_{(-)}$, $f_{(-)}$ (35) versus $e_{(+)}$, $f_{(+)}$ (38) and repeating the same steps as in Subsec. 5.1 we obtain in the special case $b = 0$ a completely analogous solution for the matching of interior "kinetic vacuum" (71)–(80) corresponding to the $(-)$ flat region of the effective scalar potential with the exterior region with the same generalized Reissner–Nordström–de Sitter geometry carrying a nonzero constant radial background electric field as in (84)–(86) with (93)–(94) upon substitution $\big(e_{(+)}, f_{(+)}\big) \to \big(e_{(-)}, f_{(-)}\big)$ and $M_{1,2} \to f_{1,2}$. In particular, instead of (95) we now obtain the following relation among the parameters of the model:

$$1 + \epsilon\frac{f_1^2}{f_2} - 3\epsilon f_0^2 e^2 = 0\,. \tag{102}$$

For the energy densities inequality instead of (101) we now have:

$$\rho_{\rm out}(r) \leq \frac{1}{4\epsilon\chi_2}\left[1 - \frac{1}{1 + \epsilon f_1^2/f_2 + \epsilon e^2 f_0^2}\right] < \rho_{\rm in} = \frac{1}{4\epsilon\chi_2}\,. \tag{103}$$

Thus, in the special case $\xi = 0$ $(b = 0)$ we obtain a gravitational bag-like configuration on the $(-)$ flat region of the effective scalar potential with the same properties as those of the gravitational bag solution in Subsec. 5.1 above — properties (i)–(ii) and (a)–(b).

One can straightforwardly extend the above solution to the general case of the parameter $b \neq 0$ (i.e. $\xi \neq 0$). However, the explicit expressions for the parameters Q and m in the exterior generalized Reissner–Nordström–de Sitter metric carrying a nonzero constant radial background electric field as well as the generalization of relation (102) among the theory's parameters become algebraically much more complicated. In particular, now the mass parameter $m \neq 0$. Moreover, in the general case of $b \neq 0$ $(\xi \neq 0)$ the vacuum value of the gauge field (71)) in the inner finite volume space region below the de Sitter horizon $(r < r_{(+)})$ is nonvanishing, thus again implying confinement dynamics of charge particles as in the outer space region beyond the de Sitter horizon ($r > r_{(-)}$ (81)) where we have nonzero constant radial background electric field $|\mathbf{E}_{\rm background}| = \frac{1}{\sqrt{2}}e_{(-)}^2 f_{(-)}$ (the counterpart of (99)). On the other hand, according to (103) the vacuum energy density $\rho_{\rm in}$ in the inner finite volume space region is larger than the energy density $\rho_{\rm out}$ in the outside region.

Therefore, in the general case $b \neq 0$ $(\xi \neq 0)$ the "vacuum-like" solution describing the matching of "kinetic vacuum" to standard constant "dilaton" vacuum in the $(-)$ flat region of the effective scalar potential is a gravitational bag-like solution which shares some of the properties of the "constituent quark" model (properties (a)–(c) in Subsec. 5.1 above), however, it does *not at all* mimic the properties (i)–(iii) of MIT bag unlike the gravitational bag-like solution of Subsec. 5.1.

6. Conclusions

In the present paper, we have constructed a new kind of gravity-matter theory coupled to a nonstandard nonlinear gauge theory with the following noteworthy features:

- Instead of the canonical Riemannian space–time volume-form (generally covariant integration measure density in terms of $\sqrt{-g}$), to construct the action of the model we are employing two different and independent non-Riemannian space–time volume-forms defined in terms of two auxiliary antisymmetric tensor gauge fields of maximal rank.

- The action of our model contains apart from the standard Einstein–Hilbert R and Maxwell gauge field $-F^2$ Lagrangian terms also scalar "dilaton" parts with a noncanonical kinetic term, as well as additional R^2 and a "square-root" Maxwell term $\sqrt{-F^2}$. The specific form of our action is dictated by the requirement of global Weyl-scale invariance.
- Solving the equations of motion for the auxiliary antisymmetric tensor gauge fields building up the two non-Riemannian space–time volume-forms introduces several arbitrary integration constants, some of them spontaneously breaking the global Weyl-scale symmetry of the initial action.
- The physical meaning of the above arbitrary integration constants is revealed within the canonical Hamiltonian formalism, namely, these integration constants turn out to be conserved Dirac-constrained canonical momenta conjugated to some of the components of the auxiliary antisymmetric tensor gauge fields of maximal rank, the latter turning out to be essentially pure gauge nonpropagating degrees of freedom.
- After passing to the physical "Einstein frame" thanks to the appearance of the above arbitrary integration constants we obtain a remarkable effective matter Lagrangian of quadratic "k-essence" type. First, the latter contains an effective scalar "dilaton" potential of a very interesting form possessing two infinitely large flat regions for large negative and large positive "dilaton" φ values. Second, all the remaining terms in the "k-essence" matter Lagrangian appear multiplied by nontrivial "dilaton"-dependent coefficient functions, including nontrivial effective gauge coupling constants running with φ.
- We study systematically the static spherically symmetric "vacuum"-like configurations corresponding to each of the flat regions of the effective scalar potential.
- First, we find two globally existing in space phases corresponding to the standard constant "dilaton" vacuum values either in either of the two infinitely large flat regions. These both phases describe confinement since in both cases the vacuum value of the nonlinear gauge field is nonzero.
- Further, we find two localized in space (in "bubbles") deconfining (confinement free) phases corresponding to the so-called "kinetic dilaton vacuum" (when the quadratic "k-essence" effective action in the Einstein frame is extremized with respect to X — the "dilaton" kinetic term), where the vacuum value of the nonlinear gauge field vanishes. In one of these deconfining phases the "dilaton" lies on the (+) flat region of the effective scalar potential and in the second one it belongs to the (−) flat region with the additional restriction on the Lagrangian parameter $b = 0$. On the other hand, in the generic case of $b \neq 0$ the "kinetic dilaton vacuum" on the (−) flat region describes a localized confining phase since the vacuum value of the nonlinear gauge field is again nonzero there.

- The localized deconfining phases inside the "bubbles" coexist with outside configurations corresponding to standard constant "dilaton" vacuums and nontrivial nonlinear gauge fields carrying nonzero constant radial background electric field. The energy density inside the "bubbles" is larger that the outside energy density. Thus, the full solution inside and outside the "bubbles" is a gravitational bag-like solution mimicking some of the basic properties of the MIT bag.[1,2]
- On the other hand, the analogy of the above gravitational bag with the MIT bag is only partial one. The present gravitational bag possess other properties (overall charge, carrying nonzero "color" flux to infinity) which resemble some of the main properties of the solitonic "constituent quark" model.[37]

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Appendix A. Canonical Hamiltonian Treatment of Gravity-Matter Theories with non-Riemannian Volume-Forms

Here, we will briefly discuss the application of the canonical Hamiltonian formalism to the new gravity-matter model based on two non-Riemannian space–time volume-forms (1). In order to elucidate the proper physical meaning of the arbitrary integration constants χ_2, M_1, M_2 (14) encountered within the Lagrangian formalism's treatment of (1) it is sufficient to concentrate only on the canonical Hamiltonian structure related to the auxiliary maximal rank antisymmetric tensor gauge fields $A_{\mu\nu\lambda}$, $B_{\mu\nu\lambda}$, $H_{\mu\nu\lambda}$ and their respective conjugate momenta.

For convenience let us introduce the following short-hand notations for the field-strengths (2), (6) of the auxiliary 3-index antisymmetric gauge fields (the dot indicating time-derivative):

$$\Phi_1(A) = \dot{A} + \partial_i A^i \,, \quad A = \frac{1}{3!}\varepsilon^{ijk} A_{ijk} \,, \quad A^i = -\frac{1}{2}\varepsilon^{ijk} A_{0jk} \,, \tag{A.1}$$

$$\Phi_2(B) = \dot{B} + \partial_i B^i \,, \quad B = \frac{1}{3!}\varepsilon^{ijk} B_{ijk} \,, \quad B^i = -\frac{1}{2}\varepsilon^{ijk} B_{0jk} \,, \tag{A.2}$$

$$\Phi(H) = \dot{H} + \partial_i H^i \,, \quad H = \frac{1}{3!}\varepsilon^{ijk} H_{ijk} \,, \quad H^i = -\frac{1}{2}\varepsilon^{ijk} H_{0jk} \,. \tag{A.3}$$

Also we will use the short-hand notation:

$$\tilde{L}^{(1)}(u,\dot{u}) \equiv R + L^{(1)} \,, \quad \tilde{L}^{(2)}(u,\dot{u}) \equiv L^{(2)} + \epsilon R^2 \,, \tag{A.4}$$

where $L^{(1,2)}$ are as in (3)–(4) and where $(u, \dot{u})$ collectively denote the set of the basic gravity-matter canonical variables $(u) = \big(g_{\mu\nu}, \varphi, A_\mu\big)$ and their respective velocities.

For the pertinent canonical momenta conjugated to (A.1)–(A.3) we have:

$$\pi_A = \tilde{L}_1(u,\dot{u})\,, \quad \pi_B = \tilde{L}^{(2)}(u,\dot{u}) + \frac{1}{\sqrt{-g}}(\dot{H} + \partial_i H^i)\,, \\ \pi_H = \frac{1}{\sqrt{-g}}(\dot{B} + \partial_i B^i)\,, \tag{A.5}$$

and

$$\pi_{A^i} = 0\,, \quad \pi_{B^i} = 0\,, \quad \pi_{H^i} = 0\,. \tag{A.6}$$

The latter imply that A^i, B^i, H^i will in fact appear as Lagrange multipliers for certain first-class Hamiltonian constraints (see Eqs. (A.10)–(A.11)). For the canonical momenta conjugated to the basic gravity-matter canonical variables we have (using last relation (A.5)):

$$p_u = (\dot{A} + \partial_i A^i)\frac{\partial}{\partial \dot{u}}\tilde{L}_1(u,\dot{u}) + \pi_H\sqrt{-g}\frac{\partial}{\partial \dot{u}}L^{(2)}(u,\dot{u})\,. \tag{A.7}$$

Now, relations (A.5) and (A.7) allow us to obtain the velocities $\dot{u}$, $\dot{A}$, $\dot{B}$, $\dot{H}$ as functions of the canonically conjugate momenta $\dot{u} = \dot{u}(u, p_u, \pi_A, \pi_B, \pi_H)$, etc. (modulo some Dirac constraints among the basic gravity-matter variables due to general coordinate and gauge invariances). Taking into account (A.5)–(A.6) (and the short-hand notations (A.1)–(A.4)) the canonical Hamiltonian corresponding to (1):

$$\mathcal{H} = p_u\dot{u} + \pi_A\dot{A} + \pi_B\dot{B} + \pi_H\dot{H} - (\dot{A} + \partial_i A^i)\tilde{L}_1(u,\dot{u}) \\ - \pi_H\sqrt{-g}\left[\tilde{L}^{(2)}(u,\dot{u}) + \frac{1}{\sqrt{-g}}(\dot{H} + \partial_i H^i)\right], \tag{A.8}$$

acquires the following form as function of the canonically conjugated variables (here $\dot{u} = \dot{u}(u, p_u, \pi_A, \pi_B, \pi_H)$):

$$\mathcal{H} = p_u\dot{u} - \pi_H\sqrt{-g}\,\tilde{L}^{(2)}(u,\dot{u}) + \sqrt{-g}\pi_H\pi_B \\ - \partial_i A^i\pi_A - \partial_i B^i\pi_B - \partial_i H^i\pi_H\,. \tag{A.9}$$

From (A.9) we deduce that indeed A^i, B^i, H^i are Lagrange multipliers for the first-class Hamiltonian constraints:

$$\partial_i\pi_A = 0 \to \pi_A = -M_1 = \text{const}\,, \tag{A.10}$$

and similarly:

$$\pi_B = -M_2 = \text{const}\,, \quad \pi_H = \chi_2 = \text{const}\,, \tag{A.11}$$

which are the canonical Hamiltonian counterparts of Lagrangian constraint equations of motion (14).

Thus, the canonical Hamiltonian treatment of (1) reveals the meaning of the auxiliary 3-index antisymmetric tensor gauge fields $A_{\mu\nu\lambda}$, $B_{\mu\nu\lambda}$, $H_{\mu\nu\lambda}$ — building blocks of the non-Riemannian space–time volume-form formulation of the modified gravity-matter model (1). Namely, the canonical momenta π_A, π_B, π_H conjugated to the "magnetic" parts A, B, H (A.1)–(A.3) of the auxiliary 3-index antisymmetric tensor gauge fields are constrained through Dirac first-class constraints (A.10)–(A.11) to be constants identified with the arbitrary integration constants χ_2, M_1, M_2 (14) arising within the Lagrangian formulation of the model. The canonical momenta π^i_A, π^i_B, π^i_H conjugated to the "electric" parts A^i, B^i, H^i (A.1)–(A.3) of the auxiliary 3-index antisymmetric tensor gauge field are vanishing (A.6) which makes the latter canonical Lagrange multipliers for the above Dirac first-class constraints.

Appendix B. "Cornell"-Type Confining Potential in Curved Space–Time

Here, we will follow the steps of the derivation in Ref. 28 of effective "Cornell"-type confining potential[44–46] between quantized charged fermions based on the general formalism[47] for quantization within the canonical Hamiltonian approach *à la* Dirac of truncated gauge and gravity theories by imposing explicitly spherical symmetry on the pertinent Lagrangian action.

In the present case the corresponding nonlinear gauge field action:

$$S = \int d^4x\sqrt{-g}\big[L(F^2) + A_\mu J^\mu\big]\,, \quad L(F^2) = -\frac{1}{4}F^2 - \frac{f_0}{2}\sqrt{-F^2} \tag{B.1}$$

yields equations of motion:

$$\begin{aligned} &\partial_\nu\big(\sqrt{-g}4L'(F^2)F^{\mu\nu}\big) + \sqrt{-g}J^\mu = 0\,,\\ &L'(F^2) = -\frac{1}{4}\left(1 - \frac{f_0}{\sqrt{-F^2}}\right), \end{aligned} \tag{B.2}$$

whose $\mu = 0$ component — the nonlinear "Gauss law" constraint equation reads:

$$\frac{1}{\sqrt{-g}}\partial_i\big(\sqrt{-g}D^i\big) = J^0\,, \quad D^i = \left(1 - \frac{f_0}{\sqrt{-F^2}}\right)F^{0i}\,, \tag{B.3}$$

with $\mathbf{D} \equiv (D^i)$ denoting the electric displacement field nonlinearly related to the electric field $\mathbf{E} \equiv \big(F^{0i}\big)$ as in the last relation (B.3).

In the special case of nonlinear gauge field theory (B.1) there exists a nontrivial vacuum solution $\sqrt{-F^2_{\rm vac}} = f_0$, which implies simultaneous vanishing of the electric displacement field, $\mathbf{D} = 0$ meaning zero observed charge, and at the same time nontrivial electric field. In particular, for static spherically symmetric fields in static spherically symmetric metric (of the form (49) with general $\mathcal{A}(r)$) the only surviving component of $F_{\mu\nu}$ is the nonvanishing radial component of the electric field $E^r = -F_{0r}$, so that $\sqrt{-F^2_{\rm vac}} = \sqrt{2}|\mathbf{E}| = f_0$. This can be viewed as the simplest classical manifestation of charge confinement: $\mathbf{D} = 0$ and nontrivial $\mathbf{E}$.

Here, we will employ the canonical Hamiltonian treatment in Ref. 47 and will truncate the nonlinear gauge field action to purely spherically symmetric fields, i.e. we will take $F_{0r} = \partial_0 A_r - \partial_r A_0$ independent of the space angles and the rest of the components of $F_{\mu\nu}$ being zero. The action of the truncated theory reads:

$$S_{\rm truncated} = \int dt \int dr\, 4\pi r^2 \left[\frac{1}{2}F_{0r}^2 - \frac{f_0}{\sqrt{2}}|F_{0r}| + A_0 J^0 + A_r J^r\right] . \tag{B.4}$$

Note that in (B.4) there is no explicit dependence on the Riemannian metric coefficient $\mathcal{A}(r) = -g_{00} = 1/g_{rr}$. It is now straightforward to apply the canonical Hamiltonian quantization procedure to (B.4) within the Dirac formalism for constrained dynamical systems (e.g. Ref. 48). Obviously, in the case of de Sitter space–time the radial coordinate r must be restricted to vary up to the de Sitter horizon radius r_H.

The canonically conjugated momenta with respect to A_0 and A_r read:

$$\Pi^0 = 0\,, \quad \Pi^r = 4\pi r^2 \left(F_{0r} - \frac{f_0}{\sqrt{2}}\right), \tag{B.5}$$

where the first one $\Pi^0 = 0$ is the standard primary Dirac constraint known in any gauge theory of Yang–Mills-type. For the density of the canonical Hamiltonian one obtains:

$$\mathcal{H} = \frac{1}{8\pi r^2}(\Pi^r)^2 + \frac{f_0}{\sqrt{2}}\Pi^r + \pi r^2 f_0^2 - A_r J^r + \Pi^r \partial_r A_0 - J^0 A_0\,. \tag{B.6}$$

Henceforth, for simplicity we will consider the case with no matter current $J^r = 0$. Time preservation of the primary constraint $\Pi^0 = 0$, i.e. $\frac{d}{dt}\Pi^0 = \{\Pi^0, \mathcal{H}\}_{\rm PB} = 0$ yields the standard secondary Dirac constraint — the "Gauss law" constraint:

$$\Phi_1(r) \equiv \partial_r \Pi^r + J^0 = 0\,. \tag{B.7}$$

Thus, one has to Dirac-canonically quantize the theory with canonical Hamiltonian:

$$H = \int dr \left[\frac{1}{8\pi r^2}(\Pi^r)^2 + \frac{f_0}{\sqrt{2}}\Pi^r + \pi r^2 f_0^2\right] \tag{B.8}$$

and with two first-class *à la* Dirac constraints $\Phi_{0,1} = 0$ ($\Phi_0 \equiv \Pi^0 = 0$ and $\Phi_1 = 0$ as in (B.7)), which have to be supplemented by two canonically conjugate gauge-fixing conditions $\chi_{0,1}$. Since A_0 and its conjugate momentum $\Pi^0 = 0$ do not mix with the rest of the canonical variables they have no impact on the pertinent *Dirac brackets* between A_r and Π^r to be promoted to quantum operator commutators upon quantization. Thus, we only need to choose an appropriate gauge fixing condition for the "Gauss law" constraint (B.7), which we can take in the form:

$$\chi_1(r) \equiv \int_{C(r)} dz^\lambda\, A_\lambda(z)\,. \tag{B.9}$$

Here $\int_{C(r)}$ is path integral along a spacelike geodesic $x^\lambda = x^\lambda(\xi)$ ending at the space–time point with radial coordinate r. In particular, for the interior de Sitter region ($r \leq r_H$) this spacelike geodesic can be taken in the form:

$$t(\xi) = t = \text{const}\,, \quad r(\xi) = r_H \sin\left(\frac{\xi}{r_H}\right), \\ 0 \leq \xi \leq \xi_{\text{fin}} \leq r_H \frac{\pi}{2}\,, \quad r(\xi_{\text{fin}}) = r\,, \tag{B.10}$$

where ξ is the de Sitter proper distance parameter, so that:

$$\chi_1(r) \equiv \int_0^r dz\, A_r(z)\,, \quad \left\{\Phi_1(r), \chi_1(r')\right\}_{\text{PB}} = \delta(r - r')\,. \tag{B.11}$$

Note that here and below $\delta(r - r')$ denotes the Dirac delta-function on the half-line (both r, $r' > 0$).

It is now straightforward to calculate the Dirac bracket between the canonically conjugate pair given by:

$$\left\{A_r(r), \Pi^r(r')\right\}_{\text{DB}} = \left\{A_r(r), \Pi^r(r')\right\}_{\text{PB}} - \iint dr''\, dr'''\left\{A_r(r), \Phi_1(r'')\right\}_{\text{PB}} \\ \times \left\{\Phi_1(r''), \chi_1(r''')\right\}^{-1}_{\text{PB}}\left\{\chi_1(r'''), \Pi^r(r')\right\}_{\text{PB}}\,, \tag{B.12}$$

by using the standard Poisson bracket $\left\{A_r(r), \Pi^r(r')\right\}_{\text{PB}} = \delta(r - r')$, which yields:

$$\left\{A_r(r), \Pi^r(r')\right\}_{\text{DB}} = 2\delta(r - r')\,. \tag{B.13}$$

Upon canonical quantization (B.13) becomes:

$$\left[\hat{\Pi}^r(r), \hat{A}_r(r')\right] = 2i\delta(r - r')\,, \quad \text{i.e.} \quad \hat{\Pi}^r(r) = -\frac{2i\delta}{\delta A_r(r)}\,. \tag{B.14}$$

Now, following Ref. 28 we consider a gauge invariant quantum state of two oppositely charged ($\pm e_0$) fermions located at $r = 0$ and $r = L$, respectively, explicitly given by:

$$|\Phi\rangle \equiv |\bar{\Psi}(L)\Psi(0)\rangle = \bar{\Psi}(L)\exp\left\{ie_0 \int_0^L dz\, A_r(z)\right\}\Psi(0)|0\rangle\,. \tag{B.15}$$

The average of the quantized canonical Hamiltonian (B.8) in this state (B.15), where now $\Pi^r(r)$ will act on $A_r(r)$ according to (B.14):

$$\langle\Phi|\hat{H}|\Phi\rangle \equiv V_{\text{eff}}(L)\,, \tag{B.16}$$

can be viewed as effective potential between the quantized fermionic pair generated by the nonlinear gauge field theory containing the "square-root" Maxwell term (B.1).

Using (B.14) one calculates:

$$\left[\hat{\Pi}^r(r), ie_0 \int_0^L dz\, A_r(z)\right] = 2e_0\theta(L - r)\,, \tag{B.17}$$

$$\left[\left[\left(\hat{\Pi}^r(r)\right)^2, ie_0 \int_0^L dz\, A_r(z)\right], ie_0 \int_0^L dz\, A_r(z)\right] = 8e_0^2\theta(L - r)\,, \tag{B.18}$$

where $\theta(r-r')$ denotes the step-function on the half-line (both r, $r'>0$). Plugging (B.17)–(B.18) into (B.16) we obtain:

$$V_{\rm eff}(L) = -\frac{e_0^2}{2\pi}\frac{1}{L} + e_0 f_0 \sqrt{2}\, L + (L\text{-independent const})\,, \tag{B.19}$$

which has precisely the form of the "Cornell" potential.[44–46]

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